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EFFECTIVE ACTIONS FOR BOSONIC TOPOLOGICAL DEFECTS

Ruth Gregory

NASA/ Fermilab Astrophysics Center, Fermi National Accelerator Laboratory
P.O.Box 500, Batavia, IL 60510, U.S.A.

ABSTRACT

We consider a gauge field theory which admits p-dimensional topological defects, expanding the equations of motion in powers of the defect thickness. In this way we derive an effective action and effective equation of motion for the defect in terms of the coordinates of the p-dimensional worldsurface defined by the history of the core of the defect.

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Introduction.

There has been some interest recently in deriving higher order terms in the action of extended objects. For instance, in string theory, Polyakov[1] suggested adding an extrinsic curvature term to the string action; other authors have investigated particles with extrinsic curvature[2], however in neither case were physical justifications presented. Following the heuristic work of Nielsen and Olesen[3] (later proved by Forster[4]), who argued that the behaviour of a vortex solution they had found was that of a Nambu string, other authors argued[5] that general topological defects had 'generalised Nambu actions'

$$S_{\rm EFF} = \int_{X^\mu(\sigma^A)} \sqrt{-\gamma}\, d^{p+1}\sigma^A. \tag{1}$$

where $X^\mu(\sigma^A)$ are the spacetime coordinates of the worldsurface, $\{\sigma^A\}$ the intrinsic coordinates of the worldsurface, and γ_{AB} the intrinsic metric. It was hoped that a more detailed examination of the equations of motion for the defect would yield the higher order terms. To our knowledge, the first step in this direction was the examination of the effective action for the Nielsen-Olesen vortex to second order in the ratio of the string width to string curvature[6]. A later study[7] of the problem showed that the original reasoning had been flawed, and that in fact there were no such correction terms. The purpose of this paper is to present a general argument for obtaining an expansion for the effective action of bosonic topological defects, and in particular to demonstrate that for strings and particles no such terms exist.

First of all, we should examine what is meant by an "effective action". Generally, topological defects can arise in field theories when the vacuum manifold of the theory is non-trivial. Specifically, a p-dimensional† topological defect can form if $\Pi_{n-p-2}(M) \neq 1$ (where n is the dimension of spacetime). Such a defect is characterised by a winding number, which is the winding number of the map from a $(n-p-2)$-sphere surrounding the defect into the vacuum manifold. The static defect is a topologically stable solution to the equations of motion of the theory, and is characterised by having translational symmetry in a $(p+1)$-hyperplane, the fields depending only on the $m = (n-p-1)$ orthogonal directions. Unless the symmetry is a global one, the energy density of the defect will be highly localised around a particular hyperplane with characteristic thickness ϵ, where ϵ^{-1} is typically of the order of the symmetry breaking scale (multiplied by the root of the self coupling constant). Clearly, ϵ is extremely small, so the question naturally arises as to whether we can approximate the motion of a general topological defect by some simple set

† where p refers to the number of spatial dimensions of the defect

of equations for a $(p+1)$-dimensional hypersurface. Therefore, we somehow want to find a way of integrating out the rapid variation of the fields perpendicular to the worldsurface, thus reducing the n-dimensional field theoretic action to a $(p+1)$-dimensional worldsurface action. This is the problem of finding an effective action.

There are essentially two approaches one could take to calculate the action. Either one expands the n-dimensional action around a known field configuration, integrating out over orthogonal directions, or, one can expand the fields and field equations in powers of thickness of the defect, using integrability conditions for the n^{th} order terms to give the effective equations of motion to order n-1. Clearly the latter method is more dependable, although more involved. The former method requires greater care for consistency. We will use both methods, mainly the former to obtain the shape of the action, and the latter to confirm the equations of motion. We start by setting up our notation and conventions before systematically expanding the action around a 'known' static solution. Finally, we derive the effective action and equations of motion for the defect up to second order in the ratio of the defect size to the extrinsic curvature of its world history.

The effective action.

Let us suppose that a p-dimensional topological defect is formed during the spontaneous symmetry breakdown of a local field theory with initial symmetry group $\mathcal{G}$. We will consider only a local theory, since only a local theory has the sharp fall off in the fields that is required by our methods. Global theories have long range Goldstone boson fields which complicate the integration off the worldsurface. (For simplicity we will take $\mathcal{G}$ to be a simple Lie group, although the more general case should be transparent.) We write $\phi(=\phi_\alpha)$ to represent the multiplet of fields transforming under $\mathcal{G}$, and $A_\mu = A_{a\mu}(x)T^a_{\alpha\beta}$ as the gauge field; thus

$$\begin{aligned} \mathcal{D}_\mu\phi &= \nabla_\mu\phi + igA_\mu\phi \\ F_{\mu\nu} &= \nabla_\mu A_\nu - \nabla_\nu A_\mu + ig[A_\mu, A_\nu], \end{aligned}$$

and we take our Lagrangian to be of the form:

$$\mathcal{L} = (\mathcal{D}^\mu\phi)^\dagger\mathcal{D}_\mu\phi - \tfrac{1}{4}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} - V(\phi^\dagger\phi); \tag{2}$$

we have taken the signature of the metric to be $(+--...)$. The equations of motion are

$$\begin{aligned} \frac{\delta S}{\delta\phi^\dagger} &= -\mathcal{D}_\mu\mathcal{D}^\mu\phi - \phi V'(\phi^\dagger\phi) = 0 \\ \frac{\delta S}{\delta A_{a\mu}} &= \mathcal{D}_\nu F^{\nu\mu}_a - ig\phi^\dagger T_a\mathcal{D}^\mu\phi + \mathrm{h.c.} = 0. \end{aligned} \tag{3}$$

We will write the solution for the static defect as $\{\phi_0, A_{0\mu}\}$ which will depend only on $\{x^i\}$, the cartesian coordinates perpendicular to the $(p+1)$-hyperplane of the defect.

$$\begin{aligned} \mathcal{D}_{0i}\mathcal{D}_{0i}\phi_0 + \phi_0 V'(\phi_0^\dagger\phi_0) = 0 \\ \mathcal{D}_{0i}F_{0aij} + ig\phi_0^\dagger T_a \mathcal{D}_{0j}\phi_0 + \text{h.c.} = 0. \end{aligned} \tag{4}$$

The first step in finding the effective action is to show that $\{\phi_0, A_{0\mu}\}$ are the solutions to the equations of motion to zeroth order in the thickness of the defect. In order to do this, we need to set up a coordinate system that is tailored to the problem at hand. Clearly, given a $(p+1)$-dimensional submanifold in spacetime, we can coordinatise it by some $\{\sigma^A\}_{A=0,\dots p}$. The worldsurface is then given by $X^\mu(\sigma^A)$, and the induced metric of this surface by

$$\gamma_{AB} = \frac{\partial X^\mu}{\partial\sigma^A}\frac{\partial X_\mu}{\partial\sigma^B}. \tag{5}$$

At each point on the world history, there exists an m-dimensional normal plane which is spanned by m normals, $\{n_i^\mu(\sigma^A)\}_{i=1\dots m}$; we choose cartesian coordinates $\{\xi^i\}$ on each normal plane to correspond with the n_i^μ. We then specify that the σ^A remain constant in these normal planes to give us a set of coordinates based on the worldsurface. These coordinates will be well defined within the extrinsic curvatures of the worldsurface. Note that in terms of the new coordinates, the metric is no longer constant, and the connection no longer vanishes. In particular, an important identity, which relates the Lie derivative of the metric to the extrinsic curvature and normal fundamental form of the worldsurface[8] is

$$\mathcal{L}_{n_i} g_{\mu\nu} = 2\nabla_{(\mu}n_{i\nu)} = 2K_{i\mu\nu} + 2\beta^j_{i(\mu}n_{j\nu)}. \tag{6}$$

This is crucial in the expansion of the action - we are not only expanding around a zeroth order field configuration, but around a zeroth order metric.

Now that we have a suitable coordinate system, to examine the zero-thickness limit we rescale our variables by a factor of $1/\epsilon$. ϵ here is taken to be a representative thickness - a gauge defect in general will have more than one thickness scale associated with it: the thickness of the scalar core, and the thickness of the gauge core. We state that the ratio between these scales remans fixed as $\epsilon \to 0$. Thus, we set $\chi^i = \xi^i/\epsilon$, and induce a corresponding rescaling of the A_i gauge fields. The metric and connection on the other hand remain unchanged, since we are changing variables rather than coordinates. Thus we see that the gauge derivative parallel to the worldsurface, $\mathcal{D}_A$, as well as the connection terms of $\mathcal{D}_i$ are now suppressed by a factor of ϵ relative to the cartesian derivative perpendicular to the worldsurface $\mathcal{D}_{0i}$. Therefore in the limit $\epsilon \to 0$, the equations reduce to the static equations (4), and the zeroth order solution is $\phi_0, A_{0\mu}$. (Strictly, we should rescale our scalar field ϕ so that all quantities are of order one, and take the zero-thickness limit in such a way that the energy of the defect remains constant. Thus, strictly it is the rescaled ϕ_0 that is the zeroth order solution, however, since we are integrating out over the defect to obtain the action, this factor is irrelevant.) With finite thickness however, ϕ_0 and A_0 need not satisfy the equations of motion, for in this case, the dependency of the metric on the orthogonal directions introduces extrinsic curvature terms via

$$\begin{aligned} \mathcal{D}_\mu \mathcal{D}^\mu \phi &= -\frac{1}{\sqrt{-g}} \mathcal{D}_{0i} \sqrt{-g} \mathcal{D}_{0i} \phi \\ &= -K_i \mathcal{D}_{0i} \phi - \mathcal{D}_{0i} \mathcal{D}_{0i} \phi \end{aligned} \tag{7}$$

with a similar expression for $\mathcal{D}_\mu F^{\mu\nu}$. Therefore (3) and (4) are not necessarily equal to first order in ϵ. We therefore expand the action in powers of ϵ.

The action of a field configuration is

$$S = \int \mathcal{L}[\phi, A_\mu] \sqrt{-g} d^n x. \tag{8}$$

Let us suppose that we have a field configuration which corresponds to a topological defect moving arbitrarily, then, provided the curvature of the worldsurface remains small compared to its thickness, we expect that the field configuration will be close to the zero-thickness limit:

$$\begin{aligned} \phi &= \phi_0 + \delta\phi \\ A_\mu &= A_{0\mu} + \delta A_\mu \end{aligned} \tag{9}$$

where $\delta\phi$ and δA_μ are at least of order ϵ, and to order ϵ satisfy the linearised perturbation equations:

$$\begin{aligned} \frac{\delta S}{\delta \phi_0} + \int \left(\frac{\delta^2 S}{\delta \phi_\alpha(y) \delta \phi} \delta \phi_\alpha(y) + \delta \phi^{\dagger\beta}(y) \frac{\delta^2 S}{\delta \phi^{\dagger\beta}(y) \delta \phi} + \delta A_{a\mu}(y) \frac{\delta^2 S}{\delta A_{a\mu}(y) \delta \phi} \right) \sqrt{-g} d^n y = 0 \\ \frac{\delta S}{\delta A_{0a\mu}} + \int \left(\frac{\delta^2 S}{\delta \phi_\alpha(y) \delta A_{a\mu}} \delta \phi_\alpha(y) + \text{h.c.} + \delta A_{b\nu}(y) \frac{\delta^2 S}{\delta A_{b\nu}(y) \delta A_{a\mu}} \right) \sqrt{-g} d^n y = 0 \end{aligned} \tag{10}$$

Here we use the notation $\frac{\delta S}{\delta \phi_0}$ to indicate a functional derivative evaluated at zeroth order only in the fields ϕ and A_μ. We will write $\frac{\delta S}{\delta \phi}\Big|_0$ to indicate evaluation completely at zeroth order, including the metric. The second order functional derivatives are always evaluated in this limit.

Thus expanding the action around the zeroth order solution

$$S = S[\phi_0, A_{0\mu}, g] + \int \left(\frac{\delta S}{\delta \phi} \delta\phi + \text{h.c.} + \frac{\delta S}{\delta A_{a\mu}} \delta A_{a\mu} \right) \sqrt{-g_0} d^n x + \frac{1}{2} \int\!\!\int \Big(\frac{\delta^2 S}{\delta\phi\delta\phi} \delta\phi\delta\phi$$
$$+ \delta\phi^\dagger \frac{\delta^2 S}{\delta\phi^\dagger \delta\phi} \delta\phi + \frac{\delta^2 S}{\delta\phi\delta A_{a\mu}} \delta\phi \delta A_{a\mu} + \text{h.c.} + \frac{\delta^2 S}{\delta A_{a\mu} \delta A_{b\nu}} \delta A_{a\mu} \delta A_{b\nu} \Big) \sqrt{-g}\sqrt{-g} d^n x d^n y$$

we see that the perturbation equations reduce this to the simpler form:

$$S = S[\phi_0, A_{0\mu}, g] + \frac{1}{2} \int \left(\frac{\delta S}{\delta \phi} \delta\phi + \text{h.c.} + \frac{\delta S}{\delta A_{a\mu}} \delta A_{a\mu} \right) \sqrt{-g_0} d^n x. \tag{11}$$

Now, in order to calculate the second term, we need to know $\delta\phi$ and δA_μ to order ϵ, i.e. the solutions to (10). Clearly these will somehow depend on the K_i, however, before trying to solve (10) we should first investigate the integrability condition. We multiply each functional derivative of S in (10) by the derivative of the corresponding field, and integrate over x to obtain:

$$-\int \frac{\delta S}{\delta \phi_0} \phi_{0,\sigma} + \text{h.c.} + \frac{\delta S}{\delta A_{0a\mu}} A_{0a\mu,\sigma} = \int\!\!\int \frac{\delta^2 S}{\delta A_{b\nu} \delta A_{a\mu}} \delta A_{b\nu} A_{0a\mu,\sigma}$$
$$+ \Big(\frac{\delta^2 S}{\delta\phi\delta\phi} \delta\phi\phi_{0,\sigma} + \delta\phi^\dagger \frac{\delta^2 S}{\delta\phi^\dagger\delta\phi} \phi_{0,\sigma} + \delta A_{a\mu} \frac{\delta^2 S}{\delta A_{a\mu}\delta\phi} \phi_{0,\sigma} + \frac{\delta^2 S}{\delta\phi\delta A_{a\mu}} \delta\phi A_{0a\mu,\sigma} + \text{h.c.} \Big) \tag{12}$$

Although this expression looks involved, note that from (7), the left hand side is

$$\int -K_i \Big(\phi^\dagger_{0,\sigma} \mathcal{D}_{0i}\phi_0 + (\mathcal{D}_{0i}\phi_0)^\dagger \phi_{0,\sigma} + A_{0aj,\sigma} F_{0aij} \Big) = -\delta^k_\sigma \int K_i \Big((\mathcal{D}_{0k}\phi_0)^\dagger \mathcal{D}_{0i}\phi_0 + \text{h.c.} + F_{0akj}F_{0aij}$$
$$+ A_{0ak}\mathcal{D}_{0j}F_{0aij} + A_{0k,j}F_{0aij} - ig[A_{0k}, A_{0j}]_a F_{0aij} \Big)$$
$$= -\int \sqrt{-\gamma} K_i M_{ik} d^{p+1}\sigma \tag{13}$$

where

$$M_{ik} = \int (\mathcal{D}_{0i}\phi_0)^\dagger \mathcal{D}_{0k}\phi_0 + \text{h.c.} + F_{0ji}F_{0jk} \;\; d^m\xi$$

is a positive definite symmetric bilinear form.

The left hand side can also be rearranged to give

$$\int d^n y \sqrt{-g} \left(\Big[\frac{\partial}{\partial y^\sigma} \frac{\delta S}{\delta \phi} \Big|_0 \Big] \delta\phi(y) + \delta\phi^\dagger(y) \frac{\partial}{\partial y^\sigma} \frac{\delta S}{\delta \phi^\dagger} \Big|_0 + \delta A_{a\mu}(y) \frac{\partial}{\partial y^\sigma} \frac{\delta S}{\delta A_{a\mu}} \Big|_0 \right) \tag{14}$$

which vanishes by virtue of the zeroth order equations of motion. Hence

$$\begin{aligned} K_i M_{ik} &= 0 \\ \Rightarrow \qquad K_i &= 0. \end{aligned} \tag{15}$$

Thus the integrability condition is $K_i = 0$, and hence $\delta\phi$, $\delta A_\mu = 0$ to this order in ϵ. Therefore we come to the possibly surprising conclusion that the action is simply

$$S = S_0[\phi_o, A_{o\mu}, g]. \tag{16}$$

We could have come to the same conclusion by examining the equations of motion associated with the zeroth order action $S[\phi_o, A_{o\mu}, g_o]$ - the Nambu action. Writing $X^\mu(\sigma^A)$ for the worldsurface spacetime coordinates as before, and D_A as the worldsurface covariant derivative‡, the Nambu equations of motion (see appendix) are merely the wave equation

$$\begin{aligned} D_A D^A X^\mu(\sigma^A) &= 0 \\ \Rightarrow \qquad K_i n_i^\mu &= 0. \end{aligned}$$

Thus the Nambu action in fact implies that $K_i = 0$. This was the flaw in the previous arguments[6]: in expanding the action around a zeroth order solution a fully consistent procedure is required, all the conclusions of the zeroth order results must be used at higher order. We exhibited both techniques of calculating the perturbation solution in order to reinforce confidence in this conclusion.

We are now left with expanding

$$S[\phi_o, A_{o\mu}, g] = \int \sqrt{-g}\mathcal{L}_g[\phi_o, A_{o\mu}] d^{p+1}\sigma d^m\xi$$

around the worldsurface. By construction, in the new coordinates $g_{ij} = \delta_{ij}$, which is independent of the ξ^i, hence $\mathcal{L}_g = \mathcal{L}_{g_0}$ and we need only expand the volume element $\sqrt{-g}$ about the worldsurface.

Therefore we have

$$\sqrt{-g} = \sqrt{-g_0} + \xi^i \partial_i \sqrt{-g_0} + \frac{1}{2}\xi^i\xi^j\partial_i\partial_j\sqrt{-g_0} + \ldots \tag{17}$$

but

$$\partial_i\sqrt{-g} = \mathcal{L}_i\sqrt{-g} = \sqrt{-g}K_i$$

‡ by which we mean the spacetime covariant derivative projected onto the worldsurface, $\frac{\partial X^\mu}{\partial \sigma^A}\nabla_\mu$, rather than ${}^{(p+1)}\nabla$.

and

$$\partial_j K_i = n_j^\mu \nabla_\mu \nabla_\nu n_i^\nu = n_j^\mu \nabla_\nu \nabla_\mu n_i^\nu = -K_j^{\mu\nu} K_{i\mu\nu}$$

implies

$$\sqrt{-g} = \sqrt{-\gamma}\{1 + \xi^i K_i + \frac{1}{2}\xi^i \xi^j (K_i K_j - K_{i\mu\nu} K_j^{\mu\nu})\}. \tag{18}$$

Clearly, upon integration, linear terms will disappear, leaving a contribution to the action of

$$S = \mu_0 \int \sqrt{-\gamma}[1 - \frac{\mu_1}{\mu_0}\epsilon^2 \ {}^{(p+1)}\mathcal{R}]\, d^{p+1}\sigma \tag{19}$$

where $\mu_0 = \int \mathcal{L}_0 d^m \xi^i$ is the energy per unit p-area of the defect, $\mu_1 = \int \xi^{i2} \mathcal{L}_0 d^m \xi^i / 2\epsilon^2$ is a constant of order unity, and we have used the Gauss Codazzi relations

$$\sum_i K_i^2 - K_{i\mu\nu}^2 = -{}^{(p+1)}\mathcal{R} \tag{20}$$

to write the action in terms of the Ricci curvature of the worldsurface.

Clearly then, for $p = 0$ this 'geometric' correction term vanishes; for $p = 1$, it is a topological constant, the Euler characteristic of the surface. Only for $p \geq 2$ does this term contribute. In this case, one can use the substitution

$$D_A D_B X^\mu = n_i^\mu K_{iAB} \tag{21}$$

in (19) to find the equations of motion for the worldsurface by varying X^μ (remembering that the metric γ_{AB} and the connection depend on X^μ). From the appendix, we see

$$K_{,i} = -2\frac{\mu_1}{\mu_0} K_{0iB}^A K_{0jC}^B K_{0jA}^C \tag{22}$$

as the second order equations of motion for the worldsurface. (In fact, the right hand side of these equations vanishes identically for $p = 0$ and 1, so we could say these were the equations for all p.)

Conclusions.

Therefore, we have shown that the second order action for a topological defect is (19):

$$S = \mu_0 \int \sqrt{-\gamma}[1 - \frac{\mu_1}{\mu_0}\epsilon^2 \ {}^{(p+1)}\mathcal{R}]\, d^{p+1}\sigma^A$$

which yields the second order equation of motion:

$$K_i = -2\epsilon^2 \frac{\mu_1}{\mu_0} K_{iB}^A K_{jC}^B K_{jA}^C$$

Therefore for $p = 0, 1$ we see that there are no second order correction terms to the action. The action for a particle is the proper length of the path, and for a string, the proper area. This might indicate a necessity for a higher order expansion, however, for such higher order terms to be important, the extrinsic curvature must be of the order of the defect size, in which case, all correction terms would be important, and we might as well analyse the full field equations. Such a situation would arise for instance at a cusp in a string trajectory.

For membranes and higher dimensional defects, the effect of (19) can be estimated by considering the subsequent motion of a p-sphere of defect released from rest. In terms of the radius $R(\tau)$ of the sphere, τ the proper time of an observer moving with the defect:

$$K_r = \frac{\ddot{R}}{\sqrt{1+\dot{R}^2}} + \frac{p}{R}\sqrt{1+\dot{R}^2}$$

$$K^B_{rA} K^C_{rB} K^A_{rC} = \Big(\frac{\ddot{R}}{\sqrt{1+\dot{R}^2}}\Big)^3 + \frac{p}{R^3}\left(1+\dot{R}^2\right)^{\frac{3}{2}}$$

Therefore, if $R_0(\tau)$ is the 'Nambu' trajectory, satisfying $K_r = 0$, the second order trajectory, $R(\tau)$ is given by

$$\frac{\ddot{R}}{\sqrt{1+\dot{R}^2}} + \frac{p}{R}\sqrt{1+\dot{R}^2} = 2\epsilon^2 \frac{\mu_1}{\mu_0}(p^3 - p)\frac{(1+\dot{R}^2)^{3/2}}{R^3}$$

Therefore $0 > \ddot{R}(\tau) > R_0(\tau)$ - thus the correction has the effect of slightly resisting the collapse of the defect when it starts to become significant - this indicates that the correction is a rigidity term. Here the approximation breaks down when $R \sim \epsilon^{2/3}$ (i.e. before the spatial radius of curvature reaches ϵ). After this point, a full field theoretic treatment would be required to investigate the behaviour of the defect (if indeed it persists as such).

It would be interesting to include the effects of supersymmetry in this calculation, but this would require finding appropriate field theories with static defect solutions that spontaneously break the required spacetime (super)symmetries - a somewhat more involved task[9]. One can take the approach of requiring an effective action to have the relevant worldsurface symmetries, (for example see ref. 10) however, this would only give the 'shape' of the action. This work shows that only by analysing the actual field theory do we get information on whether any of the terms in such an expansion are non-zero.

Another useful extension of the work would be to investigate whether one can include gravity, however, the work of Geroch and Traschen[11] in four dimensions indicates that a consistent zero-thickness limit in the general case may be problematic - and indeed we have found this to be the case.

Finally, we should remark that these results are probably applicable to a wider class of soliton solutions. For instance, we found that the action for a skyrmion was simply the action of a particle. Therefore, unlike an action with extrinsic curvature, here we cannot ascertain whether 'particles' are point like or soliton like from the macroscopic motion.

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Appendix.

Here we find the equations of motion associated with the action

$$S = \int \sqrt{-\gamma}(\mu_0 + \mu_1 \epsilon^2 \ {}^{(p+1)}\mathcal{R}) d^{p+1}\sigma. \tag{A1}$$

We first write this in terms of the worldsurface coordinates by recalling (5) and (21):

$$\begin{aligned} \gamma_{AB} &= \frac{\partial X^\mu}{\partial \sigma^A} \frac{\partial X_\mu}{\partial \sigma^B} \\ D_A D_B X^\mu &= n_i^\mu K_{iAB}, \end{aligned} \tag{A2}$$

thus

$$S = \int \sqrt{-\gamma}(\mu_0 + \mu_1 \epsilon^2 [D_A D_B X^\mu D^A D^B X_\mu - (D_A D^A X^\mu)^2]) d^{p+1}\sigma. \tag{A3}$$

In varying the action with respect to X^μ, we must remember that both the metric and the connection depend on X^μ. For the metric we have:

$$\delta\gamma_{AB} = \delta X^a_{,A} X_{a,B} + X^a_{,A} \delta X_{a,B}. \tag{A3}$$

However, we do not need to evaluate $\delta\Gamma^A_{BC}$ since this always appears multiplied by a single derivative of X^μ which is contracted with a double derivative of X^μ, a quantity perpendicular to the worldsurface. Thus

$$\begin{aligned} \delta[D_A D_B X^\mu D^A D^B X_\mu - (D_A D^A X^\mu)^2] = 2[&D_A D_B X^\mu D^A D^B \delta X_\mu - D_A D^A X^\mu D_B D^B \delta X_\mu \\ &+ \gamma^{AC}\gamma^{BD}\delta\gamma_{CD} D_A D_B X^\nu D_C D^C X_\nu] \end{aligned} \tag{A4}$$

which, upon integration by parts gives a contribution to δS of

$$\begin{aligned}\int\sqrt{-\gamma}\,2\mu_1\epsilon^2\delta X_\mu D_A\{D_B X^\mu \mathcal{R}^{AB} - D_B X^\mu D^A D^B X^\nu D_C D^C X_\nu\}\\ = \int\sqrt{-\gamma}\,2\mu_1\epsilon^2\delta X_\mu D_A\{D_B X^\mu D^A D^C X^\nu D_C D^B X_\nu\}.\end{aligned} \tag{A5}$$

Here we have used the Riemmann identity:

$$D_A D^B D^A X^\mu = D^B D_A D^A X^\mu + \mathcal{R}^{BC} D_C X^\mu \tag{A6}$$

and the Gauss-Codazzi relation

$$\begin{aligned}\mathcal{R}_{AB} &= K_{iAC}K^C_{iB} - K_i K_{iAB}\\ &= D_C D^C X^\nu D_A D_B X_\nu - D_A D_C X^\nu D^C D_B X_\nu\end{aligned} \tag{A7}$$

to simplify the equations.

Finally, we note that

$$\delta\sqrt{-\gamma} = \frac{1}{2}\gamma^{AB}\delta\gamma_{AB} = D^A X^\mu D_A \delta X_\mu \tag{A8}$$

which we may readily integrate by parts to obtain the full variation of the action as

$$\begin{aligned}\delta S = \int\sqrt{-\gamma}\delta X_\mu\Big[\mu_0 D_A D^A X^\mu + \mu_1\epsilon^2 D_A\{D^A X^\mu[(D_B D_C X^\nu)^2 - (D_B D^B X^\nu)^2]\\ + 2D_B X^\mu D^A D^C X^\nu D_C D^B X_\nu\}\Big]\end{aligned} \tag{A9}$$

Therefore, the equations of motion are

$$\begin{aligned}D_A D^A X^\mu + \frac{\mu_1}{\mu_0}\epsilon^2 D_A\Big\{D^A X^\mu[(D_B D_C X^\nu)^2 - (D_B D^B X^\nu)^2]\\ + 2D_B X^\mu D^A D^C X^\nu D_C D^B X_\nu\Big\} = 0.\end{aligned} \tag{A10}$$

Thus to zeroth order the worldsurface satisfies the wave equation:

$$\begin{aligned} & D_A D^A X_0^\mu = 0\\ \Rightarrow\quad & n_i^\mu K_i = 0.\end{aligned} \tag{A11}$$

Substituting this result into (A10) we see that to second order X^μ satisfies:

$$\left(D_A D^A X^\mu\right)_2 + \frac{2\mu_1}{\mu_0}\epsilon^2 D_B D_C X_0^\mu D^C D^A X_0^\nu D^B D_A X_{0\nu} = 0 \qquad (A12)$$

or, rewriting this in terms of the extrinsic curvatures

$$K_{i2} - \frac{2\mu_1}{\mu_0}\epsilon^2 K^A_{i0B} K^B_{j0C} K^C_{j0A} = 0. \qquad (A13)$$

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